UNIFICATION OF INDEPENDENCE IN QUANTUM PROBABILITY

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Abstract

Let $(*_{l\in I}\mathcal{A}, *_{l\in I}(\phi_l,\psi_l))$, be the conditionally free product of unital free *-algebras $\mathcal{A}_l$, where ϕ_l, ψ_l are states on $\mathcal{A}_l$, $l \in I$. We construct a sequence of noncommutative probability spaces $(\widetilde{\mathcal{A}}^{(m)}, \widetilde{\Phi}^{(m)})$, $m \in \mathbf{N}$, where $\widetilde{\mathcal{A}}^{(m)} = \bigotimes_{l\in I}\widetilde{\mathcal{A}}_l^{\otimes m}$ and $\widetilde{\Phi}^{(m)} = \bigotimes_{l\in I}\widetilde{\phi}_l \otimes \widetilde{\psi}_l^{\otimes(m-1)}$, $m \in \mathbf{N}$, $\widetilde{\mathcal{A}}_l = \mathcal{A} * \mathbf{C}[t]$, and the states $\widetilde{\phi}_l, \widetilde{\psi}_l$ are Boolean extensions of ϕ_l, ψ_l, $l \in I$, respectively. We define unital *-homomorphisms $j^{(m)} : *_{l\in I}\mathcal{A}_l \to \widetilde{\mathcal{A}}^{(m)}$ such that $\widetilde{\Phi}^{(m)} \circ j^{(m)}$ converges pointwise to $*_{l\in I}(\phi_l,\psi_l)$. Thus, the variables $j^{(m)}(w)$, where w is a word in $*_{l\in I}\mathcal{A}_l$, converge in law to the conditionally free variables. The sequence of noncommutative probability spaces $(\mathcal{A}^{(m)}, \Phi^{(m)})$, where $\mathcal{A}^{(m)} = j^{(m)}(*_{l\in I}\mathcal{A}_l)$ and $\Phi^{(m)}$ is the restriction of $\widetilde{\Phi}^{(m)}$ to $\mathcal{A}^{(m)}$, is called a *hierarchy of freeness*. Since all finite joint correlations for known examples of independence can be obtained from tensor products of appropriate *-algebras, this approach can be viewed as a unification of independence. Finally, we show how to make the m-fold free product $\widetilde{\mathcal{A}}^{*(m)}$ into a cocommutative *-bialgebra associated with m-freeness.

1. Introduction

The aim of this paper is to show that the main types of noncommutative independence can be obtained from tensor independence and are related to appropriately constructed *-bialgebras.

Since different models have led to almost separate theories and techniques, it seems desirable to develop one theory covering all the cases, including tensor, free and Boolean independence as well as their various modifications. This work makes the first step in this direction, namely is provides a unified treatment of the main notions of independence existing in the literature in the sense that it reduces the problem of calculating finite joint correlations to a similar problem formulated for the tensor product of *-algebras. In other words, it shows that the main types of products of states can be reduced to tensor products of states or they are pointwise limits of such states like in the case of freeness.

From the axiomatic approach presented in [Sch2] it follows that under certain assumptions there are three "pure" kinds of independence, namely commutative [C-H, G-vW], Boolean [vW] and free [Voi, V-D-N], each with characteristic combinatorics. An interpolation between the Boolean model and the free model has been given in terms of the so-called conditional freeness (earlier called ψ-independence) [B-S], which allows us to extract both models as special cases. Essentially, the conditionally free probability is based on the approach to the free probability presented by Voiculescu. Noncommutative probability spaces obtained from this kind of approach have always been viewed as very noncommutative and thus not directly related to the tensor product case. Our theory provides a sequence of explicit tensor product constructions which allows to (pointwise) approximate the conditionally free product of states and thus may be viewed as a unifying tensorization scheme.

In our approach, instead of making the theory noncommutative in the definition of the product of *-algebras, we stick to the tensor product and simply take noncommutative extensions of those *-algebras with non-canonical embeddings. Our ideas go back to the central limit theorem for the *-Hopf algebra $U_q(su(2))$ in [Len1, Len2]. This was our version of "q-independence", which gave the q-Gaussian law in the q-central limit theorem. It seemed natural that one should be able to construct suitable *-bialgebras associated with free independence and Boolean independence. An axiomatic approach to this subject was presented by Schürmann [Sch2]. Our approach is different and is the first one which gives explicit *-homomorphic embeddings of the free product of unital free *-algebras into suitable tensor products. Moreover, our construction can also be used for *-algebras for which $\mathcal{A}^0$ is a *-subalgebra of $\mathcal{A}$, where $\mathcal{A} = \mathcal{A}^0 \oplus \mathbf{C1}$. Thus, it is not less general than the approach in [Sch2] (see Section 3). Moreover, it gives a nice structure embodied by the constructed hierarchy of freeness and this way fills the "gap" between Boolean independence and freeness.

The main idea consists in constructing *-bialgebras (or, *-Hopf algebras, if possible) similar to the q-bialgebras or the q-deformed enveloping algebras $U_q(su(2))$, but perhaps "more noncommutative", i.e. with "more noncommutative" kernels replacing the kernels given by q-relations studied in [Sch1, Len1, Len2]. It presents no difficulty to construct a *-bialgebra associated with Boolean independence, but in order to cover freeness as well as the general case of conditional freeness, one needs to construct a sequence of *-bialgebras in order to obtain freeness as the limit in law (by which we understand the convergence of finite joint correlations).

This works for independent copies of the same algebra. If we want to consider (free [Av, Voi], conditionally free [B-L-S], Boolean [vW]) products of different algebras, a natural generalization of the *-bialgebra techniques can be used. Instead of the sequence of coproducts $\Delta^{(m)}$, we take a sequence of *-homomorphisms $j^{(m)}$ (see Definition 2.1). It turns out that in order to obtain the Boolean product it is enough to consider the 1-fold tensor product. The 2-fold tensor product construction gives a noncommutative probability space that we associate with 2-freeness, and so on, the m-fold tensor product giving m-freeness. Consequently, in the limit $m \to \infty$ we obtain freeness. In fact, all those constructions can be embedded into one, using the infinite tensor product of *-algebras, but it is convenient in some places to carry out the proofs for the sequence of m-fold tensor product constructions.

The implications of this fact should lead to some new interesting developments of the theory. It is not clear at this point to what extent our result will facilitate a unified approach to other aspects of quantum probability. It is also hard to claim that the main results in quantum probability will be reducible to the tensor product techniques and, in the case of independent copies of the same *-algebra, to the probability theory for *-bialgebras or *-Hopf algebras, no matter how nice this connection might seem. However, we think that our result provides a nice structure of independence in the noncommutative probability theory and perhaps will lead to a unified treatment of such topics as limit theorems, invariance principles, Fock spaces, etc.

In Section 2 we give basic definitions related to the extensions of states on unital free *-algebras and we introduce a sequence $(\widetilde{\mathcal{A}}^{(m)}, \widetilde{\Phi}^{(m)})$ of quantum probability spaces. Namely, for each $m \in \mathbf{N}$ and given two unital free *-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ we define the algebraic tensor product

$$\widetilde{\mathcal{A}}^{(m)} = \widetilde{\mathcal{A}}_1^{\otimes m} \otimes \widetilde{\mathcal{A}}_2^{\otimes m}$$

where $\widetilde{\mathcal{A}}_l = \mathcal{A}_l * \mathbf{C}[t]$, $l = 1, 2$, with hermitian t. Given two pairs of states on $\mathcal{A}_1, \mathcal{A}_2$, namely (ϕ_1, ψ_1) and (ϕ_2, ψ_2), respectively, we construct the tensor product state

$$\widetilde{\Phi}^{(m)} \equiv \widetilde{\Phi}_1^{(m)} \otimes \widetilde{\Phi}_2^{(m)} = \left(\widetilde{\phi}_1 \otimes \widetilde{\psi}_1^{\otimes(m-1)}\right) \otimes \left(\widetilde{\phi}_2 \otimes \widetilde{\psi}_2^{\otimes(m-1)}\right),$$

where $\widetilde{\phi}_1$, $\widetilde{\phi}_2$ are Boolean extensions of ϕ_1, ϕ_2, respectively (see Definition 2.0), to states on $\widetilde{\mathcal{A}}_l$, $l = 1, 2$. For each $m \in \mathbf{N}$ we construct a *-subalgebra $\mathcal{A}^{(m)}$ of $\widetilde{\mathcal{A}}^{(m)}$ on which the restriction of $\widetilde{\Phi}^{(m)}$ denoted by $\Phi^{(m)}$ can be interpreted as the (conditionally) m-free product state. The pair $(\mathcal{A}^{(m)}, \Phi^{(m)})$ is then the noncommutative probability space associated with m-freeness. In particular, 1-freeness is in this scheme assigned to the Boolean product and Boolean independence.

In Section 3 we prove a number of technical results, especially certain factorization lemmas which enable us to formulate our main results.

These are presented in Section 4, where we show that $\Phi^{(m)} \circ j^{(m)}$ converges pointwise to the conditionally free product of states $*_{l\in\{1,2\}}(\phi_i, \psi_i)$ on $*_{i\in\{1,2\}}\mathcal{A}_i$. In particular, when $\psi_l = \phi_l$, $l = 1, 2$, we obtain in the limit the free product of Voiculescu. We also show how our results can be extended to the case of infinitely many *-algebras. An uncountable number of free *-algebras can be treated along the same lines.

In Section 5 we restrict ourselves to the case of one unital free *-algebra: $\mathcal{A}_l = \mathcal{A}$ for all $l \in \mathbf{N}$. This corresponds to the case of conditionally free convolution powers of states on $\mathcal{A}$. For each $m \in \mathbf{N}$ we equip the m-fold free product $\widetilde{\mathcal{A}}^{*(m)} = \widetilde{\mathcal{A}} * \ldots * \widetilde{\mathcal{A}}$ (m times) with a *-bialgebra structure $(\widetilde{\mathcal{A}}^{*(m)}, \Delta^{(m)}, \epsilon^{(m)})$ with coproduct $\Delta^{(m)}$ and counit $\epsilon^{(m)}$ (in this notation the symbols of products and units are supressed), which has an interesting property. Namely, if we lift two tensor product states $\widetilde{\Phi}_1^{(m)}, \widetilde{\Phi}_2^{(m)}$ to states $\widehat{\Phi}_1$, $\widehat{\Phi}_2$, respectively, on $\widetilde{\mathcal{A}}^{*(m)}$, the convolution of $\widehat{\Phi}_1^{(m)}$ and $\widehat{\Phi}_2^{(m)}$, which by definition is expressed in terms of the coproduct as $\widehat{\Phi}_1^{(m)} \star \widehat{\Phi}_2^{(m)} \equiv (\widehat{\Phi}_1^{(m)} \otimes \widehat{\Phi}_2^{(m)}) \circ \Delta^{(m)}$, satisfies

$$\lim_{m\to\infty} \left(\widehat{\Phi}_1^{(m)} \star \widehat{\Phi}_2^{(m)}\right) \circ \widehat{\imath}_1(w) = (\phi_1, \psi_1) \star (\phi_2, \psi_2)(w)$$

where $\widehat{\imath}_1$ is the canonical *-homomorphic embedding of $\mathcal{A}$ into $\widetilde{\mathcal{A}}^{*(m)}$ given by $a \to a_{(1)}$, where $a_{(1)}$ is the first copy of the generator a in $\widetilde{\mathcal{A}}^{*(m)}$.

We view our model as a unified model of independence in the sense that finite joint correlations for known types of independence can be obtained from tensor products of appropriately defined *-algebras and tensor products of states. Here, the model of free probability of Voiculescu takes the distinguished place of a limit case. In a subsequent paper we will show that using infinite tensor products and the GNS construction one can in fact embed all levels of freeness in one tensor product of algebras. A connection with the free product representation will alos be established there.

2. Preliminaries

By a noncommutative probability space we understand a pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a unital *-algebra and $\phi : \mathcal{A} \to \mathbf{C}$ is a state, i.e. a normalized ($\phi(\mathbf{1}) = 1$), positive ($\phi(xx^*) \geq 0$ for all $x \in \mathcal{A}$) functional.

Our construction will be carried out for unital free *-algebras $\mathcal{A}$ generated by a set $\mathcal{G}^+$. We denote $\mathcal{G}^- = \{a^* | a \in \mathcal{G}^+\}$, $\mathcal{G} = \mathcal{G}^+ \cup \mathcal{G}^-$. Nonempty words in $\mathcal{A}$ will be denoted by $w = a_1 \dots a_k$, where $a_i \in \mathcal{G}$. The length of w will be denoted by $l(w)$. We allow the empty word, which is denoted by $\mathbf{1}$, of length $l(\mathbf{1}) = 0$. The involution is given by the antilinear extension of $(a_1 \dots a_k)^* = a_k^* \dots a_1^*$.

For a given unital free *-algebra $\mathcal{A}$ we consider the free product of $\mathcal{A}$ and $\mathbf{C}[t]$, the algebra of polynomials in one hermitian variable t, which we denote

$$\widetilde{\mathcal{A}} = \mathcal{A} * \mathbf{C}[t].$$

In this free product we identify units. Also, we equip $\widetilde{\mathcal{A}}$ with a natural involution defined by the antilinear extension of

$$(t_0 w_1 t_1 \dots w_n t_n)^* = t_n w_n^* \dots t_1 w_1^* t_0,$$

where $w_1, \dots, w_n$ are non-empty words in $\mathcal{A}$, and $t_0, \dots, t_n$ are monomials in $\mathbf{C}[t]$, respectively, of which $t_1, \dots, t_{n-1} \neq \mathbf{1}$.

Below we will define an extension of a state ϕ on $\mathcal{A}$ to a state $\widetilde{\phi}$ on $\widetilde{\mathcal{A}}$ which we refer to as the *Boolean extension* of ϕ.

Definition 2.0. *For a given state ϕ on $\mathcal{A}$, we define a Boolean extension of ϕ to be a functional $\widetilde{\phi}$ on $\widetilde{\mathcal{A}}$, which is the linear extension of $\widetilde{\phi}(\mathbf{1}) = 1$ and*

$$\widetilde{\phi}\,(t_0 w_1 t_1 \dots w_n t_n) = \phi(w_1) \dots \phi(w_n)$$

where $w_1, \dots, w_n$ are non-empty words in $\mathcal{A}$ and $t_0, \dots, t_n$ are words in $\mathbf{C}[t]$, of which $t_1, \dots, t_{n-1}$ are non-empty.

One can obtain $\widetilde{\phi}$ from the Boolean product of ϕ and a *-homomorphism $h : \mathbf{C}[t] \to \mathbf{C}$, for which $h(t) = 1$. In fact, from the definition of the Boolean product $\phi *_B h$ (see, for instance [B-L-S]), we obtain

$$\phi *_B h\,(t_0 w_1 t_1 \dots w_n t_n) = h(t_0) \dots h(t_n) \phi(w_1) \dots \phi(w_n)$$

and using the assumptions on h given above, we obtain the Boolean extension of ϕ.

From [B-L-S] it follows that $\widetilde{\phi}$ is a state. It is also easy to see that the two sided *-ideal generated by $t(\mathbf{1}-t)$ is contained in ker$\widetilde{\phi}$. Thus we can put $t^n = t$ and $\mathbf{1}-t = (\mathbf{1}-t)^n$ in all formulas written modulo ker $\widetilde{\phi}$. In other words, $\widetilde{\phi}$ does not distinguish between positive powers of t.

One can say that the generator t serves as a "Boolean identity", in contrast to $U_q(su(2))$-type Hopf algebras, where a similar object satisfies certain q-commutation relation and can be viewed as a "q-identity". Note that it plays the role of a "separator" of words from the *-algebra $\mathcal{A}$. This nice property will be crucial in further considerations.

Let us also recall the definition of the conditionally free product of *-algebras. For a given family of unital *-algebras $\mathcal{A}_l$, $l \in I$, and given pairs of states ϕ_l, ψ_l on $\mathcal{A}_l$, one can define a state $\phi = *_{l\in I}(\phi_l, \psi_l)$ on their free product $*_{l\in I}\mathcal{A}_l$ by $\phi(\mathbf{1}) = 1$ and the factorization property

$$\phi(a_1 \ldots a_n) = \phi_{k_1}(a_1) \ldots \phi_{k_n}(a_n),$$

whenever $a_j \in \mathcal{A}_{k_j}$ and $\psi_{k_j}(a_j) = 0$, where $k_1 \neq k_2 \neq \ldots \neq k_n$. In particular, when $\psi_j = \phi_j$, we obtain the free independence, and when $\psi_j = \pi_1$, where $\pi_1(\mathbf{1}) = 1$ and $\pi_1(w) = 0$ for any non-empty word w, we get Boolean independence.

For given two unital free *-algebras $\mathcal{A}_1, \mathcal{A}_2$ generated by $\mathcal{G}_1^+, \mathcal{G}_2^+$, respectively, let $\mathcal{G}_l = \mathcal{G}_l^+ \cup \mathcal{G}_l^-$, where $\mathcal{G}_l^- = \{a^* | a \in \mathcal{G}_l^+\}$, $l = 1, 2$. Given two pairs of states on those *-algebras, namely (ϕ_l, ψ_l), $l = 1, 2$, we construct their Boolean extensions $(\widetilde{\phi}_l, \widetilde{\psi}_l)$ on $\widetilde{\mathcal{A}}_l$ as explained in Section 2. Using them, we will construct for each $m \in \mathbf{N}$ a new noncommutative probability space $(\widetilde{\mathcal{A}}^{(m)}, \widetilde{\Phi}^{(m)})$, where

$$\widetilde{\mathcal{A}}^{(m)} = \widetilde{\mathcal{A}}_1^{\otimes m} \otimes \widetilde{\mathcal{A}}_2^{\otimes m}$$

and the state $\widetilde{\Phi}^{(m)}$ is given by

$$\widetilde{\Phi}^{(m)} = \widetilde{\phi}_1 \otimes \widetilde{\psi}_1^{\otimes(m-1)} \otimes \widetilde{\phi}_2 \otimes \widetilde{\psi}_2^{\otimes(m-1)}.$$

The involution on the $2m$-fold tensor product is given by

$$(b_1 \otimes \ldots \otimes b_m \otimes c_1 \otimes \ldots \otimes c_m)^* = b_1^* \otimes \ldots \otimes b_m^* \otimes c_1^* \otimes \ldots \otimes c_m^*.$$

Let $i_{k,m}$, $m \in \mathbf{N}, k \in [m] \equiv \{1, \ldots, m\}$ be the canonical *-homomorphic embeddings of $\widetilde{\mathcal{A}}_l$ into $\widetilde{\mathcal{A}}_l^{(m)}$ (for each l we use the same notation), i.e.

$$i_{k,m}(a) = I_{k-1} \otimes a \otimes I_{m-k}, \quad i_{k,m}(t) = I_{k-1} \otimes t \otimes I_{m-k},$$

where $a \in \mathcal{G}$, $I_k = \mathbf{1}^{\otimes k}$, extended by linearity and multiplicativity to $\widetilde{\mathcal{A}}_l$. We will adopt the convention that $i_{m+1,m}(a) = 0$. We will also use the abbreviated notation for products of $i_{k,m}(t)$'s. Namely

$$t_{[k,m]} = i_{k,m}(t) \ldots i_{m,m}(t) = I_{k-1} \otimes t^{\otimes(m-k+1)}.$$

Definition 2.1 *For given $a \in \mathcal{G}_1, b \in \mathcal{G}_2$ and $m \in \mathbf{N}$ let*

$$j_1^{(m)}(a) = \sum_{k=1}^{m} (i_{k,m}(a) - i_{k+1,m}(a)) \otimes t_{[k,m]}$$

$$j_2^{(m)}(b) = \sum_{k=1}^{m} t_{[k,m]} \otimes (i_{k,m}(b) - i_{k+1,m}(b))$$

*and define the *-homomorphism*

$$j^{(m)} : \mathcal{A}_1 * \mathcal{A}_2 \to \widetilde{\mathcal{A}}^{(m)}$$

as the linear extension of $j^{(m)}(\mathbf{1}) = I_m \otimes I_m$ *and*

$$j^{(m)}(w_1 \dots w_n) = j_{k_1}^{(m)}(w_1) \dots j_{k_n}^{(m)}(w_n),$$

where $w_1, \dots, w_n$ *are non-empty words in* $\mathcal{A}_{k_1}, \dots, \mathcal{A}_{k_n}$, *where* $k_1, \dots, k_n \in \{1, 2\}$.

Equivalently, we can write the above condition in terms of the generators, i.e.

$$j^{(m)}(a_1 \dots a_n) = j_{k_1}^{(m)}(a_1) \dots j_{k_n}^{(m)}(a_n),$$

where $a_l \in \mathcal{G}_{k_l}$, $l = 1, \dots, n$.

Remark 1. We can also write the defining relations of Definition 2.1 in the following way:

$$j_1^{(m)}(a) = \sum_{k=1}^{m} j_{1,k}^{(m)}(a), \quad j_2^{(m)}(a) = \sum_{k=1}^{m} j_{2,k}^{(m)}(a)$$

where

$$j_{1,k}^{(m)}(a) = i_{k,m}(a) \otimes (t_{[k,m]} - t_{[k-1,m]}),$$

$$j_{2,k}^{(m)}(b) = (t_{[k,m]} - t_{[k-1,m]}) \otimes i_{k,m}(b),$$

where $a \in \mathcal{G}_1$, $b \in \mathcal{G}_2$ and we understand that $t_{[0,m]} = 0$. It turns out that both ways of writing Definition 2.1 (and its generalizations introduced later) are useful, the first one – for the *-bialgebra construction, the second one – for recurrence relations. We will use them interchangably.

Remark 2. The following notation will also be used:

$$\mathcal{A}^{(m)} = j^{(m)}(\mathcal{A}_1 * \mathcal{A}_2) \quad \text{and} \quad \Phi^{(m)} = \widetilde{\Phi}^{(m)}|_{\mathcal{A}^{(m)}}.$$

Moreover, the state $\Phi^{(m)} \circ j^{(m)}$ on $\mathcal{A}_1 * \mathcal{A}_2$ will be called the *m-free product state.*

Remark 3. By $L^{(m)}$ we denote the two-sided ideal in $\widetilde{\mathcal{A}}^{(m)}$ generated by $i_{k,2m}(t(\mathbf{1} - t))$, $1 \le k \le 2m$. It is easy to see that $L^{(m)} \subset \ker \widetilde{\Phi}^{(m)}$.

Proposition 2.2. *Let* w, v *are non-empty words in* $\mathcal{A}_1, \mathcal{A}_2$, *respectively. Then*

$$j_1^{(m)}(w) = \sum_{k=1}^{m} j_{1,k}^{(m)}(w) \pmod{L^{(m)}}, \quad j_2^{(m)}(v) = \sum_{k=1}^{m} j_{2,k}^{(m)}(w) \pmod{L^{(m)}},$$

where

$$j_{1,k}^{(m)}(w) = i_{k,m}(w) \otimes (t_{[k,m]} - t_{[k-1,m]}),$$

$$j_{2,k}^{(m)}(v) = (t_{[k,m]} - t_{[k-1,m]}) \otimes i_{k,m}(v).$$

Proof. Let $a, a' \in \mathcal{G}_1$. We have

$$j_1^{(m)}(a) j_1^{(m)}(a') = \sum_{k,l=1}^{m} j_{1,k}^{(m)}(a) j_{1,l}^{(m)}(a').$$

If $1 < k < l$, then we obtain

$$j_{1,k}^{(m)}(a) j_{1,l}^{(m)}(a') = i_{k,m}(a) i_{l,m}(a') \otimes (t_{[k,m]} - t_{[k-1,m]})(t_{[l,m]} - t_{[l-1,m]})$$

$$= i_{k,m}(a) i_{l,m}(a') \otimes (I_{k-2} \otimes (\mathbf{1} - t) \otimes t^{\otimes(m-k+1)})(I_{l-2} \otimes (\mathbf{1} - t) \otimes t^{\otimes(m-l+1)})$$

$$= i_{k,m}(a) i_{l,m}(a') \otimes I_{k-2} \otimes (\mathbf{1} - t) \otimes t^{\otimes(l-k-1)} \otimes t(\mathbf{1} - t) \otimes (t^2)^{\otimes(m-l+1)} = 0 \pmod{L^{(m)}}.$$

If $1 = k < l$, then a similar analysis leads to

$$j_{1,1}^{(m)}(a) j_{1,l}^{(m)}(a') = i_{1,m}(a) i_{l,m}(a') \otimes t_{[1,m]}(t_{[l,m]} - t_{[l-1,m]})$$

$$= i_{1,m}(a) i_{l,m}(a') \otimes t^{\otimes m}(I_{l-2} \otimes (\mathbf{1} - t) \otimes t^{\otimes(m-l+1)})$$

$$= i_{1,m}(a) i_{l,m}(a') \otimes t^{\otimes l-2} \otimes t(\mathbf{1} - t) \otimes (t^2)^{\otimes(m-l+1)} = 0 \pmod{L^{(m)}}.$$

Due to commutations, the case $k > l$ does not have to be considered. Now,

$$j_{1,k}^{(m)}(a) j_{1,k}^{(m)}(a') = i_{k,m}(aa') \otimes (t_{[k,m]} - t_{[k-1,m]})(t_{[k,m]} - t_{[k-1,m]})$$

$$= i_{k,m}(aa') \otimes (I_{k-2} \otimes (\mathbf{1} - t)^2 \otimes (t^2)^{\otimes(m-k+1)}) = j_{1,k}^{(m)}(aa') \pmod{L^m}.$$

This reasoning can now be extended to a product of n generators. The proof for $j_2^{(m)}(v)$ is similar. $\square$

REMARK. Note that if we considered not free *-algebras, but *-algebras, for which in the decomposition $\mathcal{A}_k = \mathcal{A}_k^0 \oplus \mathbf{C1}$, $\mathcal{A}_k^0$ is a *-subalgebra of $\mathcal{A}_k$, then we could obtain such algebras from the associated free *-algebras by considering relations that do not involve the units. But from the above proposition it is easy to see that such relations are preserved by $j_k^{(m)}$, $k = 1, 2$ (modulo $L^{(m)}$). Therefore, our construction will also be valid for such unital *-algebras.

Before we consider the general case, we look at the simplest case first, i.e. $m = 1$. Then

$$\widetilde{\mathcal{A}}^{(1)} = \widetilde{\mathcal{A}}_1 \otimes \widetilde{\mathcal{A}}_2, \quad \widetilde{\Phi}^{(1)} = \widetilde{\phi}_1 \otimes \widetilde{\phi}_2$$

and

$$j_1^{(1)}(a) = a \otimes t, \quad j_2^{(1)}(b) = t \otimes b,$$

where a, b are generators of $\mathcal{A}_1, \mathcal{A}_2$, respectively. Thus, if w, v are nonempty words in $\mathcal{A}_1$, $\mathcal{A}_2$, respectively, then

$$j_1^{(1)}(w) = w \otimes t^{l(w)} = w \otimes t \pmod{L^{(1)}}$$

$$j_2^{(1)}(v) = t^{l(v)} \otimes v \quad = t \otimes v \pmod{L^{(1)}}$$

where $l(w), l(v)$ are the lenghts of words w, v, respectively.

We obtain for $m = 1$ the Boolean factorization law:

$$\widetilde{\Phi}^{(1)}\left(j_{k_1}^{(1)}(w_1)\ldots j_{k_n}^{(1)}(w_n)\right) = \phi_{k_1}(w_1)\ldots\phi_{k_n}(w_n)$$

with $\widetilde{\Phi}^{(1)}(\mathbf{1}) = 1$. Thus we can write

$$\widetilde{\Phi}^{(1)} \circ j^{(1)} \equiv \phi_1 *_B \phi_2,$$

where $\phi_1 *_B \phi_2$ denotes the Boolean product of ϕ_1 and ϕ_2. Thus the Boolean model is associated wih 1-freeness. This terminology can be justified by means of the following argument: $\widetilde{\Phi}^{(1)} \circ j^{(1)}$ agrees with the conditionally free product on words $w_1 w_2$. This is the trivial case but simple enough to see how the ideas of our approach developed. In the sequel we will construct succesive approximations of (conditional) freeness, using tensor products of higher orders.

Before we go on, let us write down a simple result for the Boolean case which will be used later.

Proposition 2.3. *The following factorization property holds:*

$$\widetilde{\Phi}^{(1)}\left((j_{k_1}^{(1)}(w_1) - d_{k_1}^{(1)}(w_1))\ldots(j_{k_n}^{(1)}(w_n) - d_{k_n}^{(1)}(w_n))\right)$$

$$= (\phi_{k_1}(w_1) - \psi_{k_1}(w_1))\ldots(\phi_{k_n}(w_n) - \psi_{k_n}(w_n))$$

where $w_1, \ldots, w_n$ *are non-empty words from* $\mathcal{A}_{k_1}, \ldots, \mathcal{A}_{k_n}$, $k_1 \neq k_2 \neq \ldots \neq k_n$, *and*

$$d_1^{(1)}(w) = \psi_1(w) \otimes t, \quad d_2^{(1)}(w) = t \otimes \psi_2(w).$$

Proof. This property follows directly from the fact that t plays in both $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ the role of a separator of non-empty words from $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. □

The next order of freeness will be associated with the double tensor product

$$\widetilde{\mathcal{A}}^{(2)} = \widetilde{\mathcal{A}}_1 \otimes \widetilde{\mathcal{A}}_1 \otimes \widetilde{\mathcal{A}}_2 \otimes \widetilde{\mathcal{A}}_2$$

and the double tensor product state

$$\widetilde{\Phi}^{(2)} = \widetilde{\phi}_1 \otimes \widetilde{\psi}_1 \otimes \widetilde{\phi}_2 \otimes \widetilde{\psi}_2$$

with the *-homomorphism $j^{(2)}$ defined by

$$j_1^{(2)}(a) = i_1(a) \otimes t_{[1,2]} + i_2(a) \otimes (t_{[2,2]} - t_{[1,2]})$$

$$\equiv a \otimes \mathbf{1} \otimes t \otimes t + \mathbf{1} \otimes a \otimes (\mathbf{1} - t) \otimes t$$

$$j_2^{(2)}(b) = t_{[1,2]} \otimes i_1(b) + (t_{[2,2]} - t_{[1,2]}) \otimes i_2(b)$$

$$\equiv t \otimes t \otimes b \otimes \mathbf{1} + (\mathbf{1} - t) \otimes t \otimes \mathbf{1} \otimes b$$

for generators $a \in \mathcal{G}_1,\ b \in \mathcal{G}_2$, respectively. Let us present two examples.

EXAMPLE 1. Let $a_1, a_2 \in \mathcal{G}_1$, $b \in \mathcal{G}_2$. Then

$$j_1^{(2)}(a_1)j_2^{(2)}(b)j_1^{(2)}(a_2) = (a_1 \otimes \mathbf{1} \otimes t \otimes t + \mathbf{1} \otimes a_1 \otimes (\mathbf{1}-t) \otimes t)$$

$$\times (t \otimes t \otimes b \otimes \mathbf{1} + (\mathbf{1}-t) \otimes t \otimes \mathbf{1} \otimes b)(a_2 \otimes \mathbf{1} \otimes t \otimes t + \mathbf{1} \otimes a_2 \otimes (\mathbf{1}-t) \otimes t)$$

$$= (a_1 \otimes \mathbf{1} \otimes t \otimes t)(t \otimes t \otimes b \otimes \mathbf{1} + (\mathbf{1}-t) \otimes t \otimes \mathbf{1} \otimes b)(a_2 \otimes \mathbf{1} \otimes t \otimes t) \pmod{L^{(2)}}.$$

Therefore,

$$\Phi^{(2)}\left(j_1^{(2)}(a_1)j_2^{(2)}(b)j_1^{(2)}(a_2)\right) = \widetilde{\Phi}^{(2)}(a_1ta_2 \otimes t \otimes tbt \otimes t)$$

$$+\widetilde{\Phi}^{(2)}(a_1a_2 \otimes t \otimes t \otimes tbt) - \widetilde{\Phi}^{(2)}(a_1ta_2 \otimes t \otimes t \otimes tbt)$$

$$= \phi_2(a_1)\phi_2(a_2)\phi_1(b) + \phi_2(a_1a_2)\psi_1(b) - \phi_2(a_1)\phi_2(a_2)\psi_1(b).$$

EXAMPLE 2. Let $a_1, a_2 \in \mathcal{G}_1$, $b_1, b_2 \in \mathcal{G}_2$. Then

$$j_1^{(2)}(a_1)j_2^{(2)}(b_1)j_1^{(2)}(a_2)j_2^{(2)}(b_2) = (a_1t \otimes t \otimes tb_1 \otimes t + a_1(\mathbf{1}-t) \otimes t \otimes t \otimes tb_1)$$

$$\times (a_2t \otimes t \otimes tb_2 \otimes t + t \otimes a_2t \otimes (\mathbf{1}-t)b_2 \otimes t) \pmod{L^{(2)}}$$

$$= a_1ta_2t \otimes t \otimes tb_1tb_2 \otimes t + a_1t \otimes ta_2t \otimes [tb_1b_2 \otimes t - tb_1tb_2 \otimes t]$$

$$+ [a_1a_2t \otimes t - a_1ta_2t \otimes t] \otimes tb_2 \otimes tb_1t \pmod{L^{(2)}}.$$

Therefore, we obtain

$$\Phi^{(2)}\left(j_1^{(2)}(a_1)j_2^{(2)}(b_1)j_1^{(2)}(a_2)j_2^{(2)}(b_2)\right) = \phi_1(a_1)\phi_1(a_2)\phi_2(b_1)\phi_2(b_2)$$

$$+\phi_1(a_1)\psi_1(a_2)\left[\phi_2(b_1b_2) - \phi_2(b_1)\phi_2(b_2)\right] + \left[\phi_1(a_1a_2) - \phi_1(a_1)\phi_1(a_2)\right]\phi_2(b_2)\psi_2(b_1).$$

In both examples we obtain the same expressions as if we calculated $*_{l\in\{1,2\}}(\phi_l, \psi_l)$ acting on a_1ba_2 and $a_1b_1a_2b_2$, respectively.

3. Factorization Lemmas

In this section we will derive some factorization lemmas that will be needed in the proofs of the main theorems in Section 4.

Let us define the following "condition" maps $\Psi^{(m)}$:

$$\Psi^{(m)} = \mathrm{id}^{\otimes(m-1)} \otimes \widetilde{\psi}_1 \otimes \mathrm{id}^{\otimes(m-1)} \otimes \widetilde{\psi}_2.$$

Note that

$$\widetilde{\Phi}^{(m)} = \widetilde{\Phi}^{(m-1)} \circ \Psi^{(m)} = \widetilde{\Phi}^{(m-2)} \circ \Psi^{(m-1)} \circ \Psi^{(m)} = \ldots = \widetilde{\Phi}^{(1)} \circ \Psi^{(2)} \circ \ldots \circ \Psi^{(m)}.$$

Proposition 3.0. *We can write*

$$\left(\Psi^{(m)} \circ j_1^{(m)}\right)(w) = j_1^{(m-1)}(w) + g_1^{(m-1)}(w) \pmod{L^{(m-1)}}$$

$$\left(\Psi^{(m)} \circ j_2^{(m)}\right)(v) = j_2^{(m-1)}(v) + g_2^{(m-1)}(v) \pmod{L^{(m-1)}}$$

where

$$g_1^{(m-1)}(w) = \psi_1(w)[I_{m-1} \otimes I_{m-2} \otimes (\mathbf{1} - t)],$$
$$g_2^{(m-1)}(v) = \psi_2(v)[I_{m-2} \otimes (\mathbf{1} - t) \otimes I_{m-1}],$$

and w, v are non-empty words in $\mathcal{A}_1, \mathcal{A}_2$, respectively.

Proof. It is an immediate consequence of Proposition 2.2. □.

Proposition 3.1. *The unital *-homomorphisms $j^{(m)}$ preserve the marginal laws, i.e.*

$$\widetilde{\Phi}^{(m)} \circ j_1^{(m)} = \phi_1, \quad \widetilde{\Phi}^{(m)} \circ j_2^{(m)} = \phi_2.$$

Proof. If $m = 1$, it is obvious. For $m > 1$, we obtain the result using the induction argument. Clearly, $\left(\Phi^{(m)} \circ j_k^{(m)}\right)(\mathbf{1}) = 1 = \phi_k(\mathbf{1})$, since ϕ_l, ψ_l are states on $\mathcal{A}_l$, $l = 1, 2$. Thus, assume that $w = a_1 \ldots a_n$ is a non-empty word in $\mathcal{A}_1$. Using Proposition 3.0, we obtain

$$\left(\widetilde{\Phi}^{(m)} \circ j_1^{(m)}\right)(w) = \widetilde{\Phi}^{(m-1)} \circ \Psi^{(m)} \left(j_1^{(m-1)}(w) + g_1^{(m-1)}(w)\right) = \widetilde{\Phi}^{(m-1)} \left(j_1^{(m-1)}(w)\right).$$

□

In the expressions for $j_k^{(m)}(w)$, $k = 1, 2$, there is always exactly one term with one separator t. Since it will have to be subtracted from $j_k^{(m)}(w)$, we introduce a new notation. Thus, for words w, v in $\mathcal{A}_1, \mathcal{A}_2$, respectively, let

$$d_1^{(m)}(w) = i_{m,m}(w) \otimes t_{[m,m]} \equiv I_{m-1} \otimes w \otimes I_{m-1} \otimes t,$$

$$d_2^{(m)}(v) = t_{[m,m]} \otimes i_{m,m}(v) \equiv I_{m-1} \otimes t \otimes I_{m-1} \otimes v.$$

Let us also note that

$$(\Psi^{(m)} \circ d_1^{(m)})(w) = \psi_1(w)[I_{m-1} \otimes I_{m-1}], \quad (\Psi^{(m)} \circ d_2^{(m)})(v) = \psi_2(v)[I_{m-1} \otimes I_{m-1}].$$

for words w, v in $\mathcal{A}_1, \mathcal{A}_2$, respectively. Moreover,

$$\Psi^{(m)}(j_k^{(m)}(w) - d_k^{(m)}(w)) = j_k^{(m-1)}(w) - h_k^{(m-1)}(w) \pmod{L^{(m-1)}}$$

where $k = 1, 2$ and

$$h_1^{(m-1)}(w) = \psi_1(w)[I_{m-1} \otimes I_{m-2} \otimes t], \quad h_2^{(m-1)}(w) = \psi_2(w)[I_{m-2} \otimes t \otimes I_{m-1}]$$

The main purpose of introducing the separator t is to obtain some factorizations of correlations. We present two easy lemmas.

Lemma 3.2. *Let $w_1, \dots, w_n$ be non-empty words in $\mathcal{A}_{k_1}, \dots, \mathcal{A}_{k_n}$, respectively, where $k_1 \neq k_2 \neq \dots \neq k_n$. Then $\Psi^{(m)}$ exhibits the following multiplicative property:*

$$\left(\Psi^{(m)} \circ j^{(m)}\right)(w_1 \dots w_n) = \left(\Psi^{(m)} \circ j_{k_1}^{(m)}\right)(w_1) \dots \left(\Psi^{(m)} \circ j_{k_n}^{(m)}\right)(w_n) \pmod{L^{(m-1)}}$$

Proof. The only thing to show is that all words from $\mathcal{A}_1$ appearing at the m-th site and all words from $\mathcal{A}_2$ appearing at the $2m$-th site are separated by t. But that immediately follows from Proposition 2.2 since each summand of $j_1^{(m)}(w)$, $w \in \mathcal{A}_1$, has t at the $2m$-th site and each summand of $j_2^{(m)}(v)$, $v \in \mathcal{A}_2$, has t at the m-th site. □

Lemma 3.3. *Let $w_1, \dots, w_p$ be non-empty words in $\mathcal{A}_{k_1}, \dots, \mathcal{A}_{k_n}$, respectively, where $k_1, \dots, k_n \in \{1, 2\}$ and $k_1 \neq k_2 \neq \dots \neq k_n$. Then*

$$\begin{aligned}\widetilde{\Phi}^{(m)} &\left[\left(j_{k_1}^{(m)}(w_1) - d_{k_1}^{(m)}(w_1)\right) \dots \left(j_{k_n}^{(m)}(w_n) - d_{k_n}^{(m)}(w_n)\right)\right] \\ &= \left(\phi_{k_1}(w_1) - \psi_{k_1}(w_1)\right) \dots \left(\phi_{k_n}(w_n) - \psi_{k_n}(w_n)\right).\end{aligned}$$

Proof. The case $m = 1$ is covered by Proposition 2.1. The general case follows from the induction argument. Note that $\widetilde{\Phi}^{(m)} = \widetilde{\Phi}^{(m-1)} \circ \Psi^{(m)}$ and use Lemma 3.2 to obtain

$$\begin{aligned}&\widetilde{\Phi}^{(m)} \left[\left(j_{k_1}^{(m)}(w_1) - d_{k_1}^{(m)}(w_1)\right) \dots \left(j_{k_n}^{(m)}(w_n) - d_{k_n}^{(m)}(w_n)\right)\right] \\ &= \widetilde{\Phi}^{(m-1)} \left[\left(j_{k_1}^{(m-1)}(w_1) - h_{k_1}^{(m-1)}(w_1)\right) \dots \left(j_{k_n}^{(m-1)}(w_n) - h_{k_n}^{(m-1)}(w_n)\right)\right]\end{aligned}$$

Write $\widetilde{\Phi}^{(m-1)} = \widetilde{\Phi}^{(m-2)} \circ \Psi^{(m-1)}$ and, since each $h_1^{(m-1)}(w_l)$ has t at the $2m-2$-th site and each $h_2^{(m-1)}(w_k)$ has t at the $m-1$-th site, $\Psi^{(m-1)}$ is multiplicative also on the products of the above type, hence we obtain

$$\begin{aligned}&\widetilde{\Phi}^{(m-2)} \left[\Psi^{(m-1)} \left(j_{k_1}^{(m-1)}(w_1) - h_{k_1}^{(m-1)}(w_1)\right) \dots \Psi^{(m-1)} \left(j_{k_n}^{(m-1)}(w_n) - h_{k_1}^{(m-1)}(w_n)\right)\right] \\ &= \widetilde{\Phi}^{(m-2)} \left[\Psi^{(m-1)} \left(j_{k_1}^{(m-1)}(w_1) - d_{k_1}^{(m-1)}(w_1)\right) \dots \Psi^{(m-1)} \left(j_{k_n}^{(m-1)}(w_n) - d_{k_1}^{(m-1)}(w_n)\right)\right] \\ &= \widetilde{\Phi}^{(m-1)} \left[\left(j_{k_1}^{(m-1)}(w_1) - d_{k_1}^{(m-1)}(w_1)\right) \dots \left(j_{k_n}^{(m-1)}(w_n) - d_{k_1}^{(m-1)}(w_n)\right)\right]\end{aligned}$$

where (in the first equation) we used

$$h_1^{(m-1)}(w) - d_1^{(m-1)}(w) \in \ker \Psi^{(m-1)}$$

$$h_2^{(m-1)}(v) - d_2^{(m-1)}(v) \in \ker \Psi^{(m-1)}$$

for words w, v in $\mathcal{A}_1$, $\mathcal{A}_2$, respectively. This finishes the proof. □

4. Main Theorems

We are ready to state our main result which says that the m-free product state agrees with the (conditionally) free product state on word products of not more than $2m$ words. The proof of that fact will be carried out in two steps. First, we show that m-freeness agrees with conditional freeness for products of at most $m+1$ words (Theorem 4.0). Then, we will improve that result in Theorem 4.1 and prove that in fact $m+1$ can be replaced by $2m$. As a corollary we obtain pointwise convergence of the m-free product states $\Phi^{(m)} \circ j^{(m)}$ to the conditionally free product state. Thus, the conditionally free case, in particular the free case, is obtained as a limit of m-fold tensor product constructions.

Theorem 4.0. *Let $\widetilde{\Phi}^{(m)} = \widetilde{\Phi}_1^{(m)} \otimes \widetilde{\Phi}_2^{(m)}$, $\widetilde{\Phi}_l^{(m)} = \widetilde{\phi}_l \otimes \widetilde{\psi}_l^{(m-1)}$, $l=1,2$, where $\widetilde{\phi}_l$, $\widetilde{\psi}_l$ are Boolean extensions of states ϕ_l, ψ_l on unital free *-algebras, $\mathcal{A}_i$, $i=1,2$. Then, if $n \leq m+1$, then $\widetilde{\Phi}^{(m)} \circ j^{(m)}$ agrees with the conditionally free product $*_{i\in\{1,2\}}(\phi_i, \psi_i)$ on word products $w_1 \ldots w_n$, where $w_1, \ldots, w_n \in \mathcal{A}_{k_1}, \ldots, \mathcal{A}_{k_n}$, respectively, and $k_1 \neq k_2 \neq \ldots \neq k_n$*

Proof. If $m=1$, then the result is trivial (Boolean case). So let us proceed with the induction. We have $(\Phi^{(m)} \circ j^{(m)})(\mathbf{1}) = \Phi^{(m)}(I_m \otimes I_m) = 1$. Now,

$$\begin{aligned}
&\Phi^{(m)}\left(j_{k_1}^{(m)}(w_1)\ldots j_{k_n}^{(m)}(w_n)\right)\\
&= \widetilde{\Phi}^{(m)}\left(\left(j_{k_1}^{(m)}(w_1) - d_{k_1}^{(m)}(w_1)\right)\ldots\left(j_{k_n}^{(m)}(w_n) - d_{k_n}^{(m)}(w_n)\right)\right)\\
&\quad + \sum_i \widetilde{\Phi}^{(m)}\left(j_{k_1}^{(m)}(w_1)\ldots d_{k_i}^{(m)}(w_i)\ldots j_{k_n}^{(m)}(w_n)\right)\\
&\quad - \sum_{i<l} \widetilde{\Phi}^{(m)}\left(j_{k_1}^{(m)}(w_1)\ldots d_{k_i}^{(m)}(w_i)\ldots d_{k_l}^{(m)}(w_l)\ldots j_{k_n}^{(m)}(w_n)\right)\\
&\quad + \ldots - (-1)^n \widetilde{\Phi}^{(m)}\left(d_{k_1}^{(m)}(w_1)\ldots d_{k_n}^{(m)}(w_n)\right).
\end{aligned}$$

Note that the above recurrence relation looks like the corresponding one for the conditionally free case, except that instead of numbers we have $d_{k_i}^{(m)}(w)$'s replacing $j_{k_i}^{(m)}(w)$'s at one, two, or more places. We invoke Lemma 3.2 to conclude that the first term on the right -hand side is identical to the conditionally free case. Thus what we need to prove is that in the remaining ones $d_{k_i}^{(m)}(w)$'s indeed behave like numbers. Thus the proof reduces to proving the following claim.

Claim:

$$\begin{aligned}
&\widetilde{\Phi}^{(m)}\left(j_{k_1}^{(m)}(w_1)\ldots d_{k_{i(1)}}^{(m)}(w_{i(1)})\ldots d_{k_{i(l)}}^{(m)}(w_{i(l)})\ldots j_{k_n}^{(m)}(w_n)\right)\\
&= *_{i\in\{1,2\}}(\phi_i,\psi_i)(w_1\ldots \breve{w}_{i(1)}\ldots \breve{w}_{i(l)}\ldots w_n)\psi_{k_{i(1)}}(w_{i(1)})\ldots\psi_{k_{i(l)}}(w_{i(l)}),
\end{aligned}$$

for $n \leq m+1$, where by $\breve{}$ we understand that the words with indices $i(1), \ldots, i(l)$, $1 \leq l \leq n$, are omitted. Note that the claim says that the operators $d_{k_i}^{(m)}(w)$ do behave like constants locally, i.e. if the correlation is not too long (for now, $n \leq m+1$) and that is the reason why we only have local freeness for fixed m.

The claim will be proved by induction. It trivially holds for $m = 1$ and $n \leq 2$, so assume that it holds for $m-1$. In particular, this inductive assumption implies that $\widetilde{\Phi}^{(m-1)} \circ j^{(m-1)}$ agrees with $*_{i\in\{1,2\}}(\phi_i, \psi_i)$ on products of $n \leq m$ words (Lemma 3.3 is used and the above recurrence relation for $m-1$).

Using $\widetilde{\Phi}^{(m)} = \widetilde{\Phi}^{(m-1)} \circ \Psi^{(m)}$, we obtain

$$\widetilde{\Phi}^{(m)}\left(j_{k_1}^{(m)}(w_1)\dots d_{k_{i(1)}}^{(m)}(w_{i(1)})\dots d_{k_{i(l)}}^{(m)}(w_{i(l)})\dots j_{k_n}^{(m)}(w_n)\right)$$

$$= \widetilde{\Phi}^{(m-1)}\left(\Psi^{(m)}(j_{k_1}^{(m)}(w_1))\dots \Psi^{(m)}(d_{k_{i(1)}}^{(m)}(w_{i(1)}))\dots \Psi^{(m)}(d_{k_{i(l)}}^{(m)}(w_{i(l)}))\dots \Psi^{(m)}(j_{k_n}^{(m)}(w_n))\right)$$

$$= \psi_{k_{i(1)}}(w_{i(1)})\dots \psi_{k_{i(l)}}(w_{i(l)})\widetilde{\Phi}^{(m-1)}\left(\prod_{i\in[n]\setminus\{i(1),\dots,i(l)\}}(j_{k_i}^{(m-1)}(w_i) + g_{k_i}^{(m-1)}(w_i))\right).$$

We used the muliplicativity of $\Psi^{(m)}$ (cf. Lemma 3.2), which still holds when some of the $j_{k_i}^{(m)}(w_i)$'s are replaced by $d_{k_i}^{(m)}(w_i)$'s ($d_1^{(m)}(w)$ and $d_2^{(m)}(v)$ have t at the $2m$-th and m-th tensor sites, respectively, and $k_1 \neq k_2 \neq \ldots \neq k_n$). We then used Proposition 3.0 to get the second equation.

It is enough to show that

$$\widetilde{\Phi}^{(m-1)}\left(j_{k_1}^{(m-1)}(w_1)\dots g_{k_{p(1)}}^{(m-1)}(w_{p(1)})\dots g_{k_{p(r)}}^{(m-1)}(w_{p(r)})\dots j_{k_n}^{(m-1)}(w_n)\right) = 0$$

for $n \leq m$ and arbitrary $k_1, \dots, k_n$ (note that we pulled out at least one $d_{k_i}^{(m)}(w_i)$ above, so the number of factors got smaller).

Let us now make some observations which will reduce the number of cases that need to be considered. We refer to $g_{k_{p(1)}}^{(m-1)}(w_{p(1)}), \dots, g_{k_{p(r)}}^{(m-1)}(w_{p(r)})$ in the above formula, although, for simplicity, the indices $k_{p(1)}, \dots, k_{p(r)}$ and $p(1), \dots, p(r)$ will not be used explicitly. Instead, we will refer to generic w, w' or v, nonempty words in $\mathcal{A}_1$, $\mathcal{A}_2$, respectively. Firstly, note that each element of type $g_1^{(m-1)}(w)$ commutes with $g_2^{(m-1)}(v)$, so if such elements stand next to each other, we can regroup them in any way we want. Secondly, without loss of generality we can replace $g_k^{(m-1)}(w)g_k^{(m-1)}(w')$ by $g_k^{(m-1)}(ww')$, $k = 1, 2$ (this only changes the above expression by a constant). Thirdly, it is enough to consider such configurations in which each $g_1^{(m-1)}(w)$ is sourrounded by $j_2^{(m-1)}(w')$'s and $g_2^{(m-1)}(w)$ is sourrounded by $j_1^{(m-1)}(w')$'s. For, if for instance we had $g_1^{(m-1)}(w)j_1^{(m-1)}(w')$ or $j_1^{(m-1)}(w')g_1^{(m-1)}(w)$, then at the $2m-2$-th tensor site we would get $t(\mathbf{1}-t)$, which is in $L^{(m-1)}$. A similar argument shows that $g_2^{(m-1)}(w)$ must be sourronded by $j_1^{(m-1)}(w')$'s. All this reduces the proof to configurations in which elements $g_{k_{p(i)}}^{(m-1)}(w_{p(i)})$ are sourrounded by $j_{k_{p(i)-1}}^{(m-1)}(w_{p(i)-1})$ and $j_{k_{p(i)+1}}^{(m-1)}(w_{p(i)+1})$ with $k_{p(i)-1} \neq k_{p(i)} \neq k_{p(i)+1}$. Thus it remains to tackle the configurations of this type and show that their contribution vanishes.

For that purpose we will replace each $g_1^{(m-1)}(w)$ and $g_2^{(m-1)}(v)$ by

$$\psi_1(w)I_{2m-2} - h_1^{(m-1)}(w), \quad \psi_2(v)I_{2m-2} - h_2^{(m-1)}(v),$$

respectively. Thus, to finally prove the claim it suffices to show that for $n \leq m$ we have

$$\widetilde{\Phi}^{(m-1)}\left(j_{k_1}^{(m-1)}(w_1)\dots h_{k_{s(1)}}^{(m-1)}(w_{p(1)})\dots h_{k_{s(u)}}^{(m-1)}(w_{p(r)})\dots j_{k_n}^{(m-1)}(w_n)\right)$$

$$= \psi_{k_{s(1)}}(w_{s(1)}) \ldots \psi_{k_{s(u)}}(w_{p(r)}) \widetilde{\Phi}^{(m-1)} \left(\prod_{i \in [n] \setminus \{s(1), \ldots, s(u)\}} (j_{k_i}^{(m-1)}(w_i)) \right)$$

for $k_1 \neq k_2 \neq \ldots \neq k_n$. In fact, one can take the consecutive indices different since $j_1^{(m-1)}$, $j_2^{(m-1)}$ are multiplicative and the configurations to which we reduced our proof had the property $k_1 \neq k_2 \neq \ldots \neq k_n$.

We write again $\widetilde{\Phi}^{(m-1)} = \widetilde{\Phi}^{(m-2)} \circ \Psi^{(m-1)}$ and use the factorization property of $\Psi^{(m-1)}$ on the left-hand side of the above equation since $h_1^{(m-1)}$ and $h_2^{(m-1)}$ have a non-zero power of t at the $2m-2$-th and $m-1$-th tensor site, respectively. We obtain

$$\begin{aligned}
&\widetilde{\Phi}^{(m-1)} \left(j_{k_1}^{(m-1)}(w_1) \ldots h_{k_{s(1)}}^{(m-1)}(w_{s(1)}) \ldots h_{k_{s(u)}}^{(m-1)}(w_{s(u)}) \ldots j_{k_n}^{(m-1)}(w_n) \right) \\
&\quad = \widetilde{\Phi}^{(m-2)} \Big(\Psi^{(m-1)}(j_{k_1}^{(m-1)}(w_1)) \ldots \Psi^{(m-1)}(h_{k_{s(1)}}^{(m-1)}(w_{s(1)})) \\
&\qquad \ldots \Psi^{(m-1)}(h_{k_{s(u)}}^{(m-1)}(w_{s(u)})) \ldots \Psi^{(m-1)}(j_{k_n}^{(m-1)}(w_n)) \Big) \\
&\quad = \widetilde{\Phi}^{(m-2)} \Big(\Psi^{(m-1)}(j_{k_1}^{(m-1)}(w_1)) \ldots \Psi^{(m-1)}(d_{k_{s(1)}}^{(m-1)}(w_{s(1)})) \\
&\qquad \ldots \Psi^{(m-1)}(d_{k_{s(u)}}^{(m-1)}(w_{s(u)})) \ldots \Psi^{(m-1)}(j_{k_n}^{(m-1)}(w_n)) \Big) \\
&= \widetilde{\Phi}^{(m-1)} \left(j_{k_1}^{(m-1)}(w_1) \ldots d_{k_{s(1)}}^{(m-1)}(w_{s(1)}) \ldots d_{k_{s(u)}}^{(m-1)}(w_{s(u)}) \ldots d_{k_n}^{(m-1)}(w_n) \right) \\
&\quad = \psi_{k_{s(1)}}(w_{s(1)}) \ldots \psi_{k_{s(u)}}(w_{s(u)}) \widetilde{\Phi}^{(m-1)} \left(\prod_{i \in [n] \setminus \{s(1), \ldots, s(u)\}} (j_{k_i}^{(m-1)}(w_i)) \right)
\end{aligned}$$

where, in the last equation, we used the inductive assumption of the claim. Thus, we have proved our claim. This also finishes the proof of the theorem. □

It turns out that the result of the theorem can be improved. Namely, one can show that $\widetilde{\Phi}^{(m)} \circ j^{(m)}$ agrees with the conditionally free product of states on word products $w_1 \ldots w_n$ for $n \leq 2m$. This is not surprising since already Example 2 in Section 2 showed that $\widetilde{\Phi}^{(2)} \circ j^{(2)}$ suffices to calculate a four-point correlation.

THEOREM 4.1. *$\widetilde{\Phi}^{(m)} \circ j^{(m)}$ agrees with the conditionally free product on word products $w_1 \ldots w_n$ for $n \leq 2m$, where $w_1, \ldots, w_n \in \mathcal{A}_{k_1}, \ldots, \mathcal{A}_{k_n}$, respectively, and $k_1 \neq k_2 \neq \ldots \neq k_n$.*

Proof. Let $n \leq 2m$. We know from Theorem 4.0 that $\widetilde{\Phi}^{(s)} \circ j^{(s)}$, $s = 2m-1$, agrees with the conditionally free product on word products $w_1 \ldots w_n$. We will show that $\widetilde{\Phi}^{(s)} \circ j^{(s)}$ agrees in fact with $\widetilde{\Phi}^{(m)} \circ j^{(m)}$ on such word products. By Proposition 2.2 we have

$$\widetilde{\Phi}^{(s)} \left(j_{k_1}^{(s)}(w_1) \ldots j_{k_n}^{(s)}(w_n) \right) = \sum_{m_1=1}^{s} \ldots \sum_{m_n=1}^{s} \widetilde{\Phi}^{(s)} \left(j_{k_1,m_1}^{(s)}(w_1) \ldots j_{k_n,m_n}^{(s)}(w_n) \right).$$

CLAIM. One obtains the following "pyramidal" formula:

$$\begin{aligned}
&\widetilde{\Phi}^{(s)} \left(j_{k_1}^{(s)}(w_1) \ldots j_{k_n}^{(s)}(w_n) \right) \\
&= \sum_{m_1=1}^{1} \sum_{m_2=1}^{2} \ldots \sum_{m_{n-1}=1}^{2} \sum_{m_n=1}^{1} \widetilde{\Phi}^{(s)} \left(j_{k_1,m_1}^{(s)}(w_1) \ldots j_{k_n,m_n}^{(s)}(w_n) \right).
\end{aligned}$$

Note that if $n = 2k$ is even we obtain a "pyramid" of height $k \le m$ with a flat top, and if $n = 2k-1$ is odd, then we get a "pyramid" of height $k \le m$ with a sharp top.

To prove our claim we assume that $k_1 = 1$. The proof for $k_1 = 2$ is similar. First, note that the only term that survives from the first summation corresponds to $m_1 = 1$. The reason is simple. That is the only term that does not have a compensator since it takes the form

$$j_{1,1}^{(s)}(w_1) = i_{1,s}(w_1) \otimes t_{[1,s]}.$$

The other ones look like

$$j_{1,r}^{(s)}(w_1) = i_{r,s}(w_1) \otimes (t_{[r,s]} - t_{[r-1,s]})$$

for $r > 1$ and thus have $t - \mathbf{1}$ at site $(2, r)$ which is not preceded by any non-empty words of $\mathcal{A}_2$ and thus give zero. By mirror reflection we can conclude that the same must happen at the other end of the correlation. Thus, we obtain

$$\widetilde{\Phi}^{(s)}\left(j_{k_1}^{(s)}(w_1)\dots j_{k_n}^{(s)}(w_n)\right)$$

$$= \sum_{m_2=1}^{s}\dots\sum_{m_{n-1}=1}^{s}\widetilde{\Phi}^{(s)}\left(j_{k_1,1}^{(s)}(w_1)j_{k_2,m_2}^{(s)}(w_2)\dots j_{k_{n-1},m_{n-1}}^{(s)}(w_{n-1})j_{k_n,1}^{(s)}(w_n)\right).$$

Suppose that we have already reduced our expresion to the following form

$$\widetilde{\Phi}^{(s)}\left(j_{k_1}^{(s)}(w_1)\dots j_{k_n}^{(s)}(w_n)\right)$$

$$= \sum_{m_1=1}^{1}\dots\sum_{m_l=1}^{l}\sum_{m_{l+1}=1}^{s}\dots\sum_{m_{n-l}=1}^{s}\sum_{m_{n-l+1}=1}^{l}\dots\sum_{m_n=1}^{1}\widetilde{\Phi}^{(s)}\left(j_{k_1,m_1}^{(s)}(w_1)\dots j_{k_n,m_n}^{(s)}(w_n)\right).$$

To fix attention, assume that $k_{l+1} = 1$. We will show that the terms in which $j_{k_{l+1},r}^{(s)}(w_{l+1})$ appears for $r > l+1$, give vanishing contribution. Such a term produces

$$i_{1,1}^{(s)}(w_1)(t_{[m_2,s]} - t_{[m_2-1,s]})\dots(t_{[m_l,s]} - t_{[m_l-1,s]})i_{r,s}(w_{l+1})\dots$$

in the place reserved for $\widetilde{\mathcal{A}}_1^{\otimes s}$ and

$$t_{[1,s]}i_{m_2,s}(w_2)\dots i_{m_l,s}(w_l)(t_{[r,s]} - t_{[r-1,s]})\dots$$

in the place reserved for $\widetilde{\mathcal{A}}_2^{\otimes s}$. The second expression is crucial. Namely, if $r > l+1$, then the t's produced by $j_{k_{l+1},r}^{(s)}(w_{l+1})$ appear at sites greater than $l+1$. At these sites there are no words of $\mathcal{A}_2$ preceding the t's. Therefore, the term with $t_{[r,s]}$ is compensated by the term with $t_{[r-1,s]}$. Again, the miror reflection gives a symmetric behavior on the other side. The proof for $k_{l+1} = 2$ is similar. This finishes the proof of the claim.

Thus, we finally have to show that in order to perform calculations for a pyramid of height m one can replace s by m, i.e.

$$\sum_{m_1=1}^{1}\sum_{m_2=1}^{2}\dots\sum_{m_{n-1}=1}^{2}\sum_{m_n=1}^{1}\widetilde{\Phi}^{(s)}\left(j_{k_1,m_1}^{(s)}(w_1)\dots j_{k_n,m_n}^{(s)}(w_n)\right)$$

$$= \sum_{m_1=1}^{1} \sum_{m_2=1}^{2} \cdots \sum_{m_{n-1}=1}^{2} \sum_{m_n=1}^{1} \widetilde{\Phi}^{(m)} \left(j_{k_1,m_1}^{(m)}(w_1) \ldots j_{k_n,m_n}^{(m)}(w_n) \right)$$

for $n \leq 2m$. Note that in the above sum there are no words of $\mathcal{A}_1$ or $\mathcal{A}_2$ at sites greater than m. They are only occupied by powers of t, but then $\widetilde{\Phi}^{(s)}$ sends them into 1's. Therefore, each $j_{k_i,m_i}^{(s)}(w_i)$ can be replaced by $j_{k_i,m_i}^{(m)}(w_i)$ and $\widetilde{\Phi}^{(s)}$ by $\widetilde{\Phi}^{(m)}$. This ends the proof. $\square$

COROLLARY 4.3. *$\widetilde{\Phi}^{(m)} \circ j^{(m)}$ converges pointwise to the conditionally free product od states.*

Proof. Obvious.

An extension of the construction presented above to the case of infinitely many free *-algebras is very natural. We will show how to do the construction of m-freeness, but we will not repeat the proofs since they require only minor modifications.

Let $\mathcal{A}_l$, $l \in \mathbf{N}$ be a family of unital free *-algebras generated by $\mathcal{G}_l^+$. Let $\mathcal{G}_l^- = \{a^* | a \in \mathcal{G}_l^+\}$, $\mathcal{G}_l = \mathcal{G}_l^- \cup \mathcal{G}_l^+$. As before, denote by $\widetilde{\mathcal{A}}_l = \mathcal{A}_l * \mathbf{C}[t]$ the free product of $\mathcal{A}_l$ and the algebra of polynomials in one variable t. For ach $l \in \mathbf{N}$ we identify the units of $\mathcal{A}_l$ and $\mathbf{C}[t]$ and, by abuse of notation, we denote the unit of each such product by $\mathbf{1}$. As before, extend states ϕ_l, ψ_l on $\mathcal{A}_l$ to $\widetilde{\phi}_l, \widetilde{\phi}_l$ on $\widetilde{\mathcal{A}}_l$, $l \in \mathbf{N}$. In the free product $*_{l\in\mathbf{N}}\mathcal{A}_l$ we identify the units of $\mathcal{A}_l$, $l \in \mathbf{N}$. Abusing notation, we also in this case denote the sequences of unital free *-algebras, *-homomorphisms and states by $j^{(m)}$, $\mathcal{A}^{(m)}$ and $\widetilde{\Phi}^{(m)}$, respectively.

DEFINITION 4.4. *For given $a \in \mathcal{G}_l$, let*

$$j_l^{(m)}(a) = \sum_{k=1}^{m} t_{[k,m]}^{\otimes(l-1)} \otimes (i_{k,m}(a) - i_{k+1,m}(a)) \otimes t_{[k,m]}^{\otimes\infty}$$

*where $t_{[k,m]}^{\otimes\infty} \equiv (t_{[k,m]})^{\otimes\infty}$ and $i_{m+1,m}(a) = 0$, and define the *-homomorphism*

$$j^{(m)}: \quad *_{l\in\mathbf{N}}\mathcal{A}_l \to \bigotimes_{l\in\mathbf{N}} \widetilde{\mathcal{A}}_l^{\otimes m}$$

as the linear extension of $j^{(m)}(\mathbf{1}) = I_m^{\otimes\infty}$ and

$$j^{(m)}(w_1 \ldots w_n) = j_{k_1}^{(m)}(w_1) \ldots j_{k_n}^{(m)}(w_n)$$

where $w_1, \ldots, w_n$ are non-empty words in $\mathcal{A}_{k_1}, \ldots, \mathcal{A}_{k_n}$ with $k_1, \ldots k_n \in \mathbf{N}$.

Consider the noncommutative probability space $(\widetilde{\mathcal{A}}^{(m)}, \widetilde{\Phi}^{(m)})$, where

$$\widetilde{\mathcal{A}}^{(m)} = \bigotimes_{i\in\mathbf{N}} \widetilde{\mathcal{A}}_i^{\otimes m}$$

and the state is given by

$$\widetilde{\Phi}^{(m)} = \bigotimes_{i\in\mathbf{N}} \widetilde{\Phi}_i^{(m)}, \quad \widetilde{\Phi}_i^{(m)} = \widetilde{\phi}_i \otimes \psi_i^{\otimes(m-1)}.$$

We will also use the m-th "condition" maps

$$\Psi^{(m)} = \bigotimes_{i \in \mathbf{N}} \Psi_i^{(m)}, \quad \Psi_i^{(m)} = \mathrm{id}^{\otimes(m-1)} \otimes \widetilde{\psi}_i.$$

All results of Sections 2-4 are easily generalized to the case of infinitely many *-algebras. The differences are purely technical and are omitted.

THEOREM 4.5. *$\widetilde{\Phi}^{(m)} \circ j^{(m)}$ agrees with the conditionally free product on word products $w_1 \dots w_n$ for $n \le 2m$, where $w_1, \dots, w_n \in \mathcal{A}_{k_1}, \dots, \mathcal{A}_{k_n}$, respectively, and $k_1 \neq k_2 \neq \dots \neq k_n$, $k_1, \dots, k_n \in \mathbf{N}$. Thus, $\widetilde{\Phi}^{(m)} \circ j^{(m)}$ converges pointwise to the conditionally free product of states.*

If we want to consider an uncountable number of *-algebras, we can take the continuous tensor product and proceed in a similar way.

5. Construction of the Associated *-Bialgebras

The tensor product constructions are good enough as long as we only want to study independence of certain variables. However, we also would like to associate a *-bialgebra with each kind of independence. In the case of the conditionally free independence it seems that one should be able to do that for each $m \in \mathbf{N}$ using the m-fold tensor product $\widetilde{\mathcal{A}}^{\otimes m}$. Nevertheless, it turns out that one needs to take the m-fold free product $\widetilde{\mathcal{A}}^{*(m)}$. The construction of this *-bialgebra is presented below. First, it is convenient to introduce a free version of the *-homomorphism $j^{(m)}$.

DEFINITION 5.0. *For given $a \in \mathcal{G}_1, b \in \mathcal{G}_2$, let $a_{(i)}$ and $b_{(i)}$, $i \in [m]$, denote different copies of a and b in $\widetilde{\mathcal{A}}_1^{*(m)}$ and $\widetilde{\mathcal{A}}_2^{*(m)}$, respectively, and let $t_{(i)}$, $i \in [m]$, stand for different copies of t in both products. Let*

$$\widehat{j}_1^{(m)}(a) = \sum_{k=1}^{m} (a_{(k)} - a_{(k+1)}) \otimes t_{[k,m]},$$

$$\widehat{j}_2^{(m)}(b) = \sum_{k=1}^{m} t_{[k,m]} \otimes (a_{(k)} - a_{(k+1)})$$

*where, by abuse of notation, $t_{[k,m]} = t_{(k)} \dots t_{(m)}$, and define the *-homomorphism*

$$\widehat{j}^{(m)} : \mathcal{A}_1 * \mathcal{A}_2 \to \widetilde{\mathcal{A}}_1^{*m} \otimes \widetilde{\mathcal{A}}_2^{*m}$$

as the linear extension of $\widehat{j}^{(m)}(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$ and

$$\widehat{j}^{(m)}(w_1 \dots w_n) = \widehat{j}_{k_1}^{(m)}(w_1) \dots \widehat{j}_{k_n}^{(m)}(w_n),$$

where $w_1, \dots, w_n$ are non-empty words in $\mathcal{A}_{k_1}, \dots, \mathcal{A}_{k_n}$, where $k_1, \dots, k_n \in \{1, 2\}$.

Let $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$ in the above definition. We can associate a cocommutative *-bialgebra with the pair $(\mathcal{A}, \widehat{j}^{(m)})$. Also, let $\delta : \mathcal{A} \to \mathcal{A} * \mathcal{A}$ be the *-homomorphism defined by $\delta(\mathbf{1}) = \mathbf{1}$, $\delta(a) = a^{(1)} + a^{(2)}$, where $a^{(1)}, a^{(2)}$ are different copies of $a \in \mathcal{G}$

in $\mathcal{A} * \mathcal{A}$. Note that δ maps a given a to the sum of different copies of a. Thus the moments of $\delta(a)$ in the product state are the moments of the sum of "independent", identically distributed random variables.

THEOREM 5.1. *The *-algebra $\widetilde{\mathcal{A}}^{*m}$ can be equipped with the coproduct*

$$\Delta^{(m)}(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \;\; \Delta^{(m)}(t_{(i)}) = t_{(i)} \otimes t_{(i)},$$

$$\Delta^{(m)}(a_{(k)} - a_{(k+1)}) = (a_{(k)} - a_{(k+1)}) \otimes t_{[k,m]} + t_{[k,m]} \otimes (a_{(k)} - a_{(k+1)})$$

where $k \in [m]$ and it is understood that $a_{(m+1)} = 0$, and the counit

$$\epsilon^{(m)}(t_{(k)}) = \epsilon^{(m)}(\mathbf{1}) = 1, \;\; \epsilon^{(m)}(a_{(k)}) = 0.$$

Moreover,

$$\widehat{j}^{(m)} \circ \delta = \Delta^{(m)} \circ \widehat{i}_1$$

*where $\widehat{i}_1 : \mathcal{A} \to \widetilde{\mathcal{A}}^{*(m)}$ is the canonical *-homomorphic embedding given by $\widehat{i}_1(a) = a_{(1)}$.*

Proof. Note that $t_{(k)}$, $k \in [m]$ (and thus also $t_{[k,m]}$, $k \in [m]$) are group-like and $a_{(k)} - a_{(k+1)}$, $k \in [m]$ are $t_{[k,m]}$-primitive. Thus it is easy to see that $\Delta^{(m)}$ is coassociative. Verifying that $\epsilon^{(m)}$ is the counit is also immediate. Therefore $(\widetilde{\mathcal{A}}^{*(m)}, \Delta^{(m)}, \epsilon^{(m)})$ becomes a *-bialgebra.

Now, $(\widehat{j}^{(m)} \circ \delta)(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1} = (\Delta^{(m)} \circ \widehat{i}_1)(1)$. If $a \in \mathcal{G}$, then

$$(\widehat{j}^{(m)} \circ \delta)(a) = \widehat{j}_1^{(m)}(a) + \widehat{j}_2^{(m)}(a)$$

$$= \sum_{k=1}^{m} (a_{(k)} - a_{(k+1)}) \otimes t_{[k,m]} + \sum_{k=1}^{m} t_{[k,m]} \otimes (a_{(k)} - a_{(k+1)})$$

$$= \Delta^{(m)} \Big((a_{(1)} - a_{(2)}) + \ldots + (a_{(m-1)} - a_{(m)}) + a_{(m)} \Big) = \Delta^{(m)}(a_{(1)}) = \Delta^{(m)} \circ \widehat{i}_1(a).$$

This implies that this identity holds also for arbitrary words in $\mathcal{A}$ since $\widehat{j}^{(m)}, \Delta^{(m)}, \delta$ and $\widehat{i}_1$ are *-homomorphisms. $\square$

The above theorem shows a relation between $\widehat{j}^{(m)} \circ \delta$ and the coproduct $\Delta^{(m)}$. Namely, $\widehat{j}^{(m)} \circ \delta$ equals the coproduct $\Delta^{(m)}$ when restricted to the *-subalgebra $\widehat{i}_1(\mathcal{A})$. However, $\Delta^{(m)}$ takes $\widehat{i}_1(\mathcal{A})$ out of $\widehat{i}_1(\mathcal{A}) \otimes \widehat{i}_1(\mathcal{A})$. That is why we had to take a bigger *-bialgebra. Moreover, looking at $\widehat{j}^{(m)} \circ \delta(a)$, we can see that the m-fold free product $\widetilde{\mathcal{A}}^{*(m)}$ is the right choice if we do not assume any additional relations between different copies of the generators of $\widetilde{\mathcal{A}}$. If we do, we can take the quotient of $\widetilde{\mathcal{A}}$ modulo a two-sided ideal providing it is also a coideal. However, many relations which appear in the tensor product construction are not preserved by the coproduct. Thus, $\Delta^{(m)}$ preserves $t_{(i)}t_{(j)} = t_{(j)}t_{(i)}$ for any i, j, and $a_{(i)}t_{(j)} = t_{(j)}a_{(i)}$ for $j < i$, but it does not preserve $a_{(i)}t_{(j)} = t_{(j)}a_{(i)}$ for $j > i$, or $a_{(i)}a'_{(j)} = a'_{(j)}a_{(i)}$ for $i \neq j$. In other words, the two-sided ideal $\mathcal{T}$ generated by all those relations is not a coideal. This is the reason why we cannot take $\widetilde{\mathcal{A}}^{\otimes m}$ and we have to stick to $\widetilde{\mathcal{A}}^{*(m)}$ or its quotient $\widetilde{\mathcal{A}}^{*(m)}/\mathcal{T}_0$, where $\mathcal{T}_0$ is the two sided ideal (and a coideal) generated by $t_{(i)}t_{(j)} = t_{(j)}t_{(i)}$. All the results can be formulated for either of these two cases (we choose $\widetilde{\mathcal{A}}^{*(m)}$).

Definition 5.2. *Let $m, N \in \mathbf{N}$. For given $a \in \mathcal{G}_i$, $i \in [N]$ let*

$$\widehat{\jmath}_{i,N}^{(m)}(a) = \sum_{k=1}^{m} t_{[k,m]}^{\otimes(i-1)} \otimes (a_{(k)} - a_{(k+1)}) \otimes t_{[k,m]}^{\otimes(N-i)},$$

*and define the *-homomorphism*

$$\widehat{\jmath}_N^{(m)} : \mathcal{A}_1 * \ldots * \mathcal{A}_N \to \bigotimes_{i=1}^{N} \widetilde{\mathcal{A}}_i^{*(m)}$$

as the linear extension of $\widehat{\jmath}_N^{(m)}(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$ and

$$\widehat{\jmath}_N^{(m)}(w_1 \ldots w_n) = \widehat{\jmath}_{k_1,N}^{(m)}(w_1) \ldots \widehat{\jmath}_{k_n,N}^{(m)}(w_n),$$

where $w_1, \ldots, w_n$ are non-empty words in $\mathcal{A}_{k_1}, \ldots, \mathcal{A}_{k_n}$, $k_1, \ldots, k_n \in [N]$ and $t_{[k,m]}^{\otimes l} \equiv (t_{[k,m]})^{\otimes l}$.

Then we can express the iterations of the coproduct in terms of $\widehat{\jmath}_N^{(m)}$ in the following way.

Corollary 5.3. *Let δ_N be the N-th iteration of δ, i.e. $\delta_N : \mathcal{A} \to \mathcal{A} * \ldots * \mathcal{A}$ (N times) is the *-homomorphism defined by $\delta_N(\mathbf{1}) = \mathbf{1}$, $\delta_N(a) = a^{(1)} + \ldots + a^{(N)}$, where $a^{(1)}, \ldots, a^{(N)}$ are different copies of a. Then*

$$\widehat{\jmath}^{(m)} \circ \delta_N = \Delta_{N-1}^{(m)} \circ \widehat{\imath}_1,$$

where the $N-1$-th iteration of the coproduct $\Delta^{(m)}$ is obtained from the recursive formula: $\Delta_1^{(m)} = \Delta^{(m)}$, $\Delta_k^{(m)} = (\mathrm{id} \otimes \Delta_{k-1}^{(m)}) \circ \Delta^{(m)}$, $k > 1$.

Proof. Clearly,

$$(\widehat{\jmath}_N^{(m)} \circ \delta_N)(\mathbf{1}) = \mathbf{1}^{\otimes N} = (\Delta_{N-1}^{(m)} \circ \widehat{\imath}_1)(\mathbf{1}).$$

Now, let $a \in \mathcal{G}$. Using $\widehat{\jmath}^{(m)}$, we obtain

$$(\widehat{\jmath}_N^{(m)} \circ \delta_N)(a) = \widehat{\jmath}^{(m)}(a^{(1)} + \ldots + a^{(N)}) = \sum_{i=1}^{N} \widehat{\jmath}_{i,N}^{(m)}(a)$$

$$= \sum_{i=1}^{N} \sum_{k=1}^{m} t_{[k,m]}^{\otimes(i-1)} \otimes (a_{(k)} - a_{(k+1)}) \otimes t_{[k,m]}^{\otimes(N-i)}$$

$$= \sum_{k=1}^{m} \Delta_{N-1}^{(m)}(a_{(k)} - a_{(k+1)}) = \Delta_{N-1}^{(m)}(a_{(1)}) = (\Delta_{N-1}^{(m)} \circ \widehat{\imath}_1)(a).$$

Thus $\widehat{\jmath}_N^{(m)} \circ \delta_N$ and $\Delta_{N-1}^{(m)} \circ \widehat{\imath}_1$ agree on the unit and the generators of $\mathcal{A}$. This is enough since δ_N, $\widehat{\jmath}_N^{(m)}$, $\Delta_{N-1}^{(m)}$ and $\widehat{\imath}_1$ are *-homomorphisms. □

Proposition 5.4. *Let $\bar{\mathcal{A}} = \mathcal{A} * \mathbf{C}[t, t^{-1}]$. The *-bialgebra $(\widetilde{\mathcal{A}}^{*(m)}, \Delta^{(m)}, \epsilon^{(m)})$ can be embedded in the *-Hopf algebra $(\bar{\mathcal{A}}^{*(m)}, \bar{\Delta}^{(m)}, \bar{\epsilon}^{(m)}, S^{(m)})$, where the coproduct $\bar{\Delta}^{(m)}$ and the counit $\bar{\epsilon}^{(m)}$ agree with $\Delta^{(m)}$ and $\epsilon^{(m)}$ on $\widetilde{\mathcal{A}}^{*(m)}$, respectively, and*

$$\bar{\Delta}^{(m)}(t_{(k)}^{-1}) = t_{(k)}^{-1} \otimes t_{(k)}^{-1}, \quad \bar{\epsilon}^{(m)}(t_{(k)}^{-1}) = 1,$$

with the antipode $S^{(m)}$ *defined by*

$$S^{(m)}(\mathbf{1}) = \mathbf{1}, \quad S^{(m)}(t_{(k)}) = t_{(k)}^{-1}, \quad S^{(m)}(t_{(k)}^{-1}) = t_{(k)},$$

$$S^{(m)}(a_{(k)} - a_{(k+1)}) = -t_{[k,m]}^{-1}(a_{(k)} - a_{(k+1)})t_{[k,m]}^{-1}.$$

Proof. Recalling the definition of a Hopf *-algebra [Kas], we need the involution and the antipode to satisfy the following conditions: (i) * is an antimorphism of real algebras as well as a morphism of real coalgebras, (ii) $S^{(m)}(S^{(m)}(x^*)^*) = x$ for all $x \in \bar{\mathcal{A}}^{*(m)}$.

The involution * is an antimorphism of real algebras by definition: $(x_1 \dots x_n)^* = x_n^* \dots x_1^*$. To show that * is a morphism of real coalgebras, one needs $\bar{\epsilon}^{(m)}$ to be a hermitian functional and the coproduct to satisfy $(* \otimes *) \circ \bar{\Delta}^{(m)} = \bar{\Delta}^{(m)} \circ *$. The first property follows from the definition of the counit. Checking the second property for the generators is immediate. This is enough since

$$(* \otimes *) \circ \bar{\Delta}^{(m)}(x_1 \dots x_k) = \bar{\Delta}^{(m)}(x_k)^* \dots \bar{\Delta}^{(m)}(x_1)^*$$

$$= \bar{\Delta}^{(m)}(x_k^*) \dots \bar{\Delta}^{(m)}(x_1^*) = \bar{\Delta}^{(m)}(x_k^* \dots x_1^*) = \bar{\Delta}^{(m)}((x_1 \dots x_k)^*).$$

Finally, property (ii) can be easily verified for generators, from which it follows that it holds for any $x \in \bar{\mathcal{A}}^{*(m)}$. □

Let us look now at the convolutions of states. It is known how to define convolutions of states for *-bialgebras. Namely, if Γ_2, Γ_1 are two states on a *-bialgebra $(\mathcal{B}, \Delta, \epsilon)$, then the convolution of Γ_1 and Γ_2 is given by

$$\Gamma_1 \star \Gamma_2 \equiv (\Gamma_1 \otimes \Gamma_2) \circ \Delta.$$

Thus, in the case of m-freeness we can express the convolution of states on $\widetilde{\mathcal{A}}^{*(m)}$ in terms of the coproduct $\Delta^{(m)}$. Let $\mathcal{T}$ be the two-sided ideal in $\widetilde{\mathcal{A}}^{*(m)}$ generated by

$$t_{(i)}t_{(j)} - t_{(j)}t_{(i)}, \quad t_{(i)}a_{(j)} - a_{(j)}t_{(i)}, \quad a_{(i)}a_{(j)} - a_{(j)}a_{(i)},$$

where $i, j \in [m]$, and $i \neq j$. Then the quotient algebra $\widetilde{\mathcal{A}}^{*(m)}/\mathcal{T}$ is canonically isomorphic to $\widetilde{\mathcal{A}}^{\otimes m}$. Denote by $\eta : \widetilde{\mathcal{A}}^{*(m)} \to \widetilde{\mathcal{A}}^{\otimes m}$ the canonical mapping. Then, for a given state $\widetilde{\Phi}$ on $\widetilde{\mathcal{A}}^{\otimes m}$, let $\widehat{\Phi} = \widetilde{\Phi} \circ \eta$. We arrive at the following corollary.

Corollary 5.5. *Let* $\widetilde{\Phi}_l^{(m)} = \widetilde{\phi}_l \otimes \widetilde{\psi}_l^{\otimes(m-1)}$, *where* ϕ_l, ψ_l *are states on* $\mathcal{A}$, *and let* $\widehat{\Phi}_l^{(m)} = \widetilde{\Phi}_l^{(m)} \circ \eta$, $l \in [2]$. *Then*

$$\lim_{m\to\infty} (\widehat{\Phi}_1^{(m)} \star \widehat{\Phi}_2^{(m)}) \circ \widehat{i}_1 = (\phi_1, \psi_1) \star (\phi_2, \psi_2)$$

pointwise, where $(\phi_1, \psi_1) \star (\phi_2, \psi_2) = *_{i\in\{1,2\}}(\phi_i, \psi_i) \circ \delta$ *is the conditionally free convolution of states.*

Proof. Let w be a word in $\mathcal{A}$. From Theorems 4.0-4.1 and Proposition 5.1 we obtain

$$\lim_{m\to\infty} (\widehat{\Phi}_1^{(m)} \star \widehat{\Phi}_2^{(m)}) \circ \widehat{i}_1(w) = \lim_{m\to\infty} (\widehat{\Phi}_1^{(m)} \otimes \widehat{\Phi}_2^{(m)}) \circ \Delta^{(m)} \circ \widehat{i}_1(w)$$

$$= \lim_{m\to\infty} (\widehat{\Phi}_1^{(m)} \otimes \widehat{\Phi}_2^{(m)}) \circ \widehat{j}^{(m)} \circ \delta(w) = *_{i\in\{1,2\}}(\phi_i, \psi_i) \circ \delta(w) = (\phi_1, \psi_1) \star (\phi_2, \psi_2)(w).$$

$\square$

Corollary 5.6. *Let* $\widetilde{\Phi}_l^{(m)} = \widetilde{\phi}_l \otimes \widetilde{\psi}_l^{\otimes(m-1)}$*, where* ϕ_l, ψ_l *are states on* $\mathcal{A}$*, and let* $\widehat{\Phi}_l^{(m)} = \widetilde{\Phi}_l^{(m)} \circ \eta$*,* $l \in [N]$*. Then*

$$\lim_{m\to\infty} (\widehat{\Phi}_1^{(m)} \star \ldots \star \widehat{\Phi}_N^{(m)}) \circ \widehat{i}_1 = (\phi_1, \psi_1) \star \ldots \star (\phi_N, \psi_N)$$

pointwise.

Proof. In the proof of Corollary 5.5 replace δ by δ_N, $\Delta^{(m)}$ by $\Delta_{N-1}^{(m)}$ and instead of tensor products and convolutions of two objects take tensor products and convolutions, respectively, of N objects. $\square$

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